$$J_1(z)=\frac{2}{\pi}\int_0^{\pi/2}\sin(z\cos\theta)\cos\theta\,d\theta$$

Thus, with the initial values

$$y_1(0)=0 \tag{6a}$$

$$y_2(0)=0,\quad y_2'(0)=\frac{1}{\sqrt{\gamma}(1+\sqrt{\gamma})} \tag{6b}$$

we find that

$$y_1(z)=\int_0^z J_0(t)\,dt \tag{7}$$

$$y_2(z)=\frac{\sqrt{(1-\gamma)}}{\sqrt{\gamma}(1+\sqrt{\gamma})}\sin\frac{z}{\sqrt{(1-\gamma)}}+\frac{1}{\sqrt{(1-\gamma)}}\int_0^z\sin\frac{z-t}{\sqrt{(1-\gamma)}}J_1(t)\,dt \tag{8}$$

Finally, after integrating the last term in Eq. (8) by parts,

$$\frac{M}{Lrc_b}=\int_0^z J_0(t)\,dt-\sqrt{\frac{\gamma}{(1-\gamma)}}\sin\frac{z}{\sqrt{1-\gamma}}+\frac{\gamma}{1-\gamma}\int_0^z\cos\frac{z-t}{\sqrt{1-\gamma}}J_0(t)\,dt \tag{9}$$

Integrals of the type in Eq. (7), which can be expressed in terms of Bessel and Struve functions,[2] are available in trabulated form.[3] Those in Eq. (9) are known as Schwartz integrals, and are discussed in some detail elsewhere.[4-6]

For values of γ in the vicinity of unity, a computationally more tractible form can be obtained by expanding the denominator of Eq. (3), and using the following formula (obtained from Poissons integral representation[2] by differentiation):

$$\int_0^{\pi/2}\sin(z\cos\theta)\sin^{2n}\theta\cos\theta\,d\theta=\frac{\pi}{2}\frac{1\cdot3\ldots(2n-1)}{z^n}J_{n+1}(z)$$

The resulting expression is

$$\frac{M}{Lrc_b}=\int_0^z J_0(t)\,dt-J_1(z)+\frac{(1-\gamma)}{\gamma}\frac{J_2(z)}{z}-1\cdot3\left[\frac{1-\gamma}{\gamma}\right]^2\frac{J_3(z)}{z^2}+\ldots+$$

Acknowledgment

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Emittance of Semi-Infinite Scattering Medium with Refractive Index Greater than Unity

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Introduction

RECENTLY, radiant transport in liquids and solids[1] has been a problem of great interest. Several investigators[2-5] have considered the influence of the refractive index on the emission characteristics of a scattering medium. Turner and Love used the time-consuming Monte Carlo technique to study a one-dimensional planar layer[2] and a two-dimensional semi-infinite planar slab.[3] The exponential kernel approximation was used to analyze a semi-infinite[4] and a finite[5] medium. All of these studies assume isotropic scattering and an isothermal medium. The objective of the present investigation is to formulate the directional emittance of a semi-infinite medium in terms of Chandrasekhar's H-function and to obtain exact numerical results for a wide range of albedos and refractive indexes.

Formulation

The system chosen for this study is a semi-infinite planar medium at a uniform temperature which scatters isotropically and is characterized by a single scattering albedo ω_ν (ratio of scatter σ_ν to extinction β_ν coefficient) and a refractive index n_ν. The subscript ν refers to the frequency under consideration. For a semi-infinite medium with incident intensity, $I_\nu^+(0,\mu)$, the source function S_ν is given by

$$S(\tau_\nu)=(1-\omega_\nu)n_\nu^2 I_{b\nu}(T)+\frac{\omega_\nu}{2}\int_0^1 I_\nu^+(0,\mu)\exp(-\tau_\nu/\mu)\,d\mu+\frac{\omega_\nu}{2}\int_0^\infty S_\nu(t)E_1(|\tau_\nu-t|)\,dt \tag{1}$$

where $I_{b\nu}$ is the Planck blackbody function corresponding to the temperature of the medium (T), $\tau_\nu=\int_0^x\beta_\nu dx$ is optical depth into the medium, $\mu=\cos\theta$ is cosine of the polar angle, and $E_1(t)=\int_0^1\exp(-t/\mu)(d\mu/\mu)$. Since integral Eq. (1) is linear, superposition can be applied and the source function expressed in terms of two universal functions, i.e.,

$$S_\nu(\tau_\nu)=(1-\omega_\nu)n_\nu^2 I_{b\nu}(T)B_s(\tau_\nu)+\frac{\omega_\nu}{2}\int_0^1 I_\nu^+(0,\mu)B(\tau_\nu,\mu)\,d\mu \tag{2}$$

where

$$B_s(\tau)=1+\frac{\omega}{2}\int_0^\infty B_s(t)E_1(|\tau-t|)\,dt \tag{3}$$

and

$$B(\tau,\mu)=\exp(-\tau/\mu)+\frac{\omega}{2}\int_0^\infty B(t,\mu)E_1(|\tau-t|)\,dt \tag{4}$$

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Physically, B_s corresponds to the dimensionless source function for an isothermal semi-infinite medium, while B corresponds to the dimensionless source function for a semi-infinite medium exposed to collimated radiation. These two functions are simply related mathematically, i.e., $B_s(\tau) = B(\tau, \infty)$.

The difficulty with Eq. (2) is that boundary condition $I_\nu^+(0, \mu)$ is unknown. Since there is no external radiation incidence on the interface, the intensity $I_\nu^+(0, \mu)$ is only the result of back reflection and is given by

$$I_\nu^+(0, \mu) = \rho_\nu(\mu) I_\nu^-(0,\mu)$$

$$= \rho_\nu(\mu) \int_0^\infty S_\nu(t) \exp(-t/\mu)(\mathrm{d}t/\mu) \tag{5}$$

where $\rho_\nu(\mu)$ is the interface reflectance. Substitution of Eq. (2) into Eq. (5) yields

$$I_\nu^+(0,\mu) = (1-\omega_\nu) n_\nu^2 I_{b\nu}(T) \rho_\nu(\mu)$$
$$\times \int_0^\infty B_s(t) \exp(-t/\mu)(\mathrm{d}t/\mu) + \frac{\omega_\nu}{2} \nu\rho_\nu(\mu) \int_0^1 I_\nu^+(0,\mu')$$
$$\times \int_0^\infty B(t,\mu') \exp(-t/\mu)\mathrm{d}t(\mathrm{d}\mu'/\mu) \tag{6}$$

Chandrasekhar has shown that [6]

$$\int_0^\infty B(t,\mu_0) \exp(-t/\mu)\mathrm{d}t = \mu\mu_0 H(\mu) H(\mu_0)/(\mu+\mu_0) \tag{7}$$

and with the identity $H(\infty) = 1/\sqrt{1-\omega}$

$$\int_0^\infty B_s(t) \exp(-t/\mu)\mathrm{d}t = \mu H(\mu)/\sqrt{1-\omega} \tag{8}$$

The famous Chandrasekhar's H-function depends on both μ and ω. Substitution of Eqs. (7) and (8) into Eq. (6) yields

$$I_\nu^+(0,\mu) = \rho_\nu(\mu)\sqrt{1-\omega_\nu} H(\mu) n_\nu^2 I_{b\nu}(T)$$
$$+ \frac{\omega_\nu}{2} \rho_\nu(\mu) \int_0^1 I_\nu^+(0,\mu') \mu' H(\mu) H(\mu') \mathrm{d}\mu'/(\mu+\mu') \tag{9}$$

and with the aid of Eq. (5)

$$I_\nu^-(0,\mu) = \sqrt{1-\omega_\nu} H(\mu) n_\nu^2 I_{b\nu}(T)$$
$$+ \frac{\omega_\nu}{2} \nu \int_0^1 \rho_\nu(\mu') I_\nu^-(0,\mu') \mu' H(\mu) H(\mu') \mathrm{d}\mu'/(\mu+\mu') \tag{10}$$

Thus, $I_\nu^+(0,\mu)$ or $I_\nu^-(0,\mu)$ satisfies a linear integral equation. Physically, the first term in Eq. (10) accounts for emission and the second accounts for backscattering of the radiation reflected at the interface. Equation (10) can be rewritten as

$$I_\nu^-(0,\mu) = \epsilon_m(\mu) n_\nu^2 I_{b\nu}(T)$$
$$+ 2\int_0^1 \rho_\nu(\mu') I_\nu^-(0,\mu') \rho_m(\mu,\mu') \mu' \mathrm{d}\mu' \tag{11}$$

where $\epsilon_m(\mu)$ and $\rho_m(\mu,\mu')$ are the directional emittance[7] and the bidirectional reflectance,[8] respectively, of a semi-infinite medium with a refractive index of unity.

The directional emittance at $\mu_1 = \cos\theta_1$, as measured on the outside of the interface, is defined by

$$\epsilon_\nu(\mu_1) = [1-\rho_\nu(\mu)] I_\nu^-(0,\mu)/[n_\nu^2 I_{b\nu}(T)] \tag{12}$$

where $\mu = [1-(1-\mu_1^2)/n^2]^{1/2}$ represents the direction in which the intensity within the medium must hit the interface in order to appear at the vacuum side in the direction μ_1 (Snell's law). The reflection characteristics of the smooth interface are governed by Fresnel relations, i.e.,

$$\rho_\nu(\mu) = \frac{1}{2}\left(\frac{\mu-n\mu_1}{\mu+n\mu_1}\right)^2 + \frac{1}{2}\left(\frac{n\mu-\mu_1}{n\mu+\mu_1}\right)^2 \tag{13}$$

where $\mu_1 = 1-n^2(1-\mu^2)^{1/2}$. At all angles greater than the critical angle $[\mu_c = (n^2-1)^{1/2}/n]$, the reflectance is unity. These relations are for a dielectric material (imaginary component of refractive index is essentially zero) and for uniformly polarized radiation.

Numerical Procedure

Since integral equation, Eq. (10), is linear, $I_\nu^-(0,\mu)$ can be expressed in terms of a universal function $f(\mu)$, i.e.,

$$I_\nu^-(0,\mu) = n_\nu^2 I_{b\nu}(T) f(\mu) \tag{14}$$

where

$$f(\mu) = \sqrt{1-\omega} H(\mu)$$
$$+ \frac{\omega}{2} \int_0^1 \rho(\mu') f(\mu') \mu' H(\mu) H(\mu') \mathrm{d}\mu'/(\mu+\mu') \tag{15}$$

Thus, the directional emittance is simply

$$\epsilon_\nu(\mu_1) = [1-\rho_\nu(\mu)] f(\mu) \tag{16}$$

Integral equation, Eq. (15), is solved by successive approximations.

The integral term is evaluated by breaking the interval [0,1] into three subintervals $[0,\mu_c]$, $[\mu_c, \mu_c + 0.05(1-\mu_c)]$, $[\mu_c + 0.05(1-\mu_c), 1]$ and by applying a 9th order Gaussian quadrature to each subinterval. The iterative process is terminated when the maximum difference between the results of two successive iterations is less than $0.5(10)^{-8}$. In general, the number of iterations increases with the albedo and the refractive index. Very accurate results can be obtained without subdividing the interval and with fewer iterations. The H-function is calculated from the following integral equation[6]:

$$\frac{1}{H(\mu)} = \sqrt{1-\omega} + \frac{\omega}{2} \int_0^1 \frac{\mu' H(\mu')}{\mu+\mu'} \mathrm{d}\mu' \tag{17}$$

using successive approximations. The hemispherical emittance is calculated from $\epsilon_\nu = 2\int_0^1 \epsilon_\nu(\mu_1)\mu_1 \mathrm{d}\mu_1$.

Results

For a nonscattering medium ($\omega = 0$), the directional emittance of the medium is governed by Fresnel relations, i.e., $\epsilon(\mu_1) = 1-\rho(\mu)$. The normal emittance is $4n/(n+1)^2$. Between a refractive index of 1.2 and 2.5, the directional emittance shows the same behavior and could be approximated by the results for $n = 1.3$. For a scattering medium ($0 < \omega < 1$), the directional character of the emittance can also be approximated by the Fresnel relation, except for n near unity and μ near zero. The error increases as μ approaches zero and as n and ω approach unity. This approximation is accurate to within 15% for $n > 1.3$ and $\omega < 0.9999$.

Table 1 Normal and hemispherical emittances

ω	$n=1.0$	$n=1.1$	$n=1.2$	$n=1.3$	$n=4/3$	$n=1.4$	$n=1.5$	$n=1.6$	$n=1.7$	$n=1.8$	$n=1.9$	$n=2.0$
						Normal						
0.00	1.0000	0.99773	0.99174	0.98299	0.97959	0.97222	0.96000	0.94675	0.93279	0.91837	0.90369	0.88889
0.10	0.98361	0.98527	0.98181	0.97490	0.97201	0.96553	0.95439	0.94200	0.92873	0.91488	0.90066	0.88626
0.30	0.94278	0.95326	0.95594	0.95362	0.95201	0.94779	0.93945	0.92930	0.91785	0.90549	0.89252	0.87915
0.50	0.88477	0.90555	0.91643	0.92058	0.92082	0.91992	0.91575	0.90901	0.90037	0.89035	0.87933	0.86760
0.60	0.84459	0.87113	0.88724	0.89576	8.89729	0.89872	0.89756	0.89332	0.88677	0.87851	0.86897	0.85850
0.70	0.79132	0.82396	0.84634	0.86043	0.86363	0.86818	0.87109	0.87031	0.86671	0.86094	0.85353	0.84489
0.80	0.71475	0.75340	0.78334	0.80475	0.81026	0.81918	0.82802	0.83242	0.83334	0.83150	0.82749	0.82179
0.85	0.66034	0.70153	0.73569	0.75167	0.76869	0.78055	0.79355	0.80173	0.80601	0.80716	0.80579	0.80242
0.900	0.58505	0.62768	0.66600	0.69720	0.70606	0.72162	0.74008	0.75344	0.76251	0.76803	0.77060	0.77076
0.925	0.53345	0.57585	0.61586	0.64976	0.65965	0.67740	0.69927	0.71603	0.72839	0.73698	0.74242	0.74520
0.950	0.46446	0.50512	0.54587	0.58211	0.59303	0.61310	0.63888	0.65982	0.67640	0.68912	0.69849	0.70498
0.975	0.35908	0.39447	0.43291	0.46948	0.48099	0.50283	0.53246	0.55825	0.58030	0.59882	0.61408	0.62638
0.990	0.24728	0.27417	0.30565	0.33759	0.34808	0.36863	0.39802	0.42531	0.45027	0.47278	0.49283	0.51047
0.995	0.18295	0.20381	0.22919	0.25587	0.26485	0.28273	0.30908	0.33447	0.45858	0.38118	0.40214	0.42138
0.999	0.08715	0.09770	0.11125	0.12623	0.13144	0.14210	0.15854	0.17529	0.19214	0.20893	0.22551	0.24176
0.9995	0.06259	0.07027	0.08026	0.09145	0.09538	0.10346	0.11608	0.12912	0.14246	0.15597	0.16955	0.18310
0.9999	0.02858	0.03215	0.03688	0.04227	0.04418	0.04816	0.05446	0.06112	0.06809	0.07532	0.08277	0.09042
1.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
						Hemisphere						
0.00	1.00000	0.97484	0.95572	0.93887	0.93354	0.92319	0.90822	0.89375	0.87968	0.86595	0.85252	0.83940
0.10	0.97830	0.96051	0.94494	0.93039	0.92566	0.91633	0.90256	0.88902	0.87567	0.86252	0.84957	0.83683
0.30	0.92555	0.92408	0.91705	0.90819	0.90497	0.89823	0.88755	0.87640	0.86494	0.85331	0.84161	0.82991
0.50	0.85346	0.87076	0.87497	0.87403	0.87296	0.86999	0.86388	0.85633	0.84778	0.83852	0.82877	0.81869
0.60	0.80528	0.83295	0.84424	0.84860	0.84901	0.84866	0.84581	0.84089	0.83448	0.82699	0.81872	0.80988
0.70	0.74344	0.78197	0.80167	0.81269	0.81503	0.81813	0.81966	0.81835	0.81494	0.80995	0.80378	0.79673
0.80	0.65813	0.70733	0.73705	0.75675	0.76168	0.76959	0.77741	0.78146	0.78260	0.78151	0.77868	0.77450
0.85	0.59983	0.65362	0.68889	0.71393	0.72055	0.73165	0.74384	0.75173	0.75625	0.75811	0.75785	9.75592
0.900	0.52198	0.57869	0.61945	0.65053	0.65917	0.67425	0.69211	0.70523	0.71449	0.72062	0.72418	0.72564
0.925	0.47037	0.52711	0.57015	0.60434	0.61411	0.63151	0.65287	0.66940	0.68188	0.69100	0.69730	0.70125
0.950	0.40333	0.45800	0.50219	0.53909	0.54997	0.56981	0.59515	0.61581	0.63240	0.64548	0.65553	0.66299
0.975	0.30500	0.35259	0.39446	0.43188	0.44340	0.46508	0.49427	0.51964	0.54146	0.56001	0.57559	0.58849
0.990	0.20544	0.24143	0.27566	0.30832	0.31881	0.33918	0.36804	0.39474	0.41919	0.44137	0.46130	0.47906
0.995	0.15015	0.17795	0.20549	0.23270	0.24166	0.25935	0.28515	0.30989	0.33338	0.35548	0.37611	0.39519
0.9990	0.07029	0.08426	0.09888	0.11408	0.11926	0.12975	0.14576	0.16198	0.17828	0.19454	0.21065	0.22650
0.9995	0.05025	0.06041	0.07117	0.08251	0.08641	0.09435	0.10662	0.11923	0.13211	0.14516	0.15832	0.17150
0.9999	0.02281	0.02752	0.03261	0.03805	0.03995	0.04385	0.04997	0.05639	0.06310	0.07006	0.07726	0.08466
1.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

The normal and hemispherical emittances for a wide range of refractive indexes and albedos are presented in Table 1. Inspection of this table reveals that the emittance increases, reaches a maximum, and then decreases with refractive index. The refractive index at which the maximum emittance occurs increases with albedo, being unity for $\omega=0$. Both the normal and hemispherical emittances are very sensitive to small amounts of absorption ($\omega \simeq 1$). The approximate solution presented by Armaly, Lam, and Crosbie[4] is accurate to within 5% for $n\leq 2$ and $\omega\leq 0.99$.

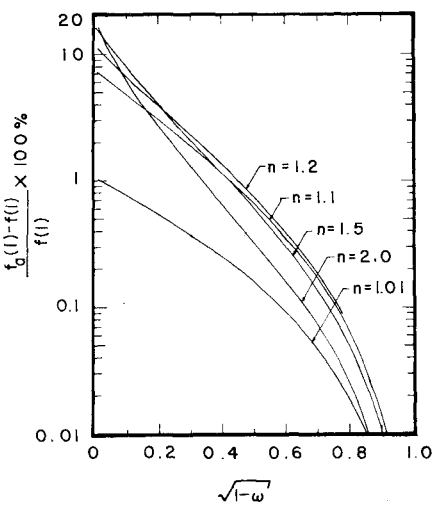

Fig. 1 Percentage error for approximation, Eq. (18).

A close inspection of the numerical results reveals that $f(\mu)$ or $I_\nu(0,\mu)$ is a weak function of μ. This is especially true for $1\leq\mu\leq\mu_c$. Employing the first term of Taylor expansion of $f(\mu')$ about $\mu'=\mu$, i.e., $f(\mu')=f(\mu)$, in Eq. (15) yields the following approximation:

$$f_a(\mu)=\sqrt{1-\omega}H(\mu)/\left[1-\frac{\omega}{2}H(\mu)\right.$$
$$\left.\times\int_0^1 \rho(\mu')\mu' H(\mu')\,d\mu'/(\mu+\mu')\right] \quad (18)$$

The validity of this approximation is illustrated in Fig. 1. The largest error occurs for small amounts of absorption ($\omega \simeq 1$). For small ω or large n, the error is small. As n approaches unity, $\rho(\mu)$ approaches zero, and thus the back reflection term in Eq. (15) approaches zero. As n become large, μ_c approaches unity and the critical angle approaches zero. Approximation, Eq. (18), overestimates the directional and hemispherical emittances and yields the largest error for the normal emittance. Thus, this approximation is accurate to within 17% for $\omega\leq 0.9999$ or within 8% for $\omega\leq 0.99$ for the entire range of refractive index.

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Selective Optimal Orthogonalization of Measured Modes

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IN a Comment on Targoff's paper,[1] Rodden[2] argues that a method of orthogonalization which does not "corrupt" the rigid body modes[3] and assigns a higher credibility to the measurements of lower frequency modes[4] is preferable over a method in which all modes are treated equally. In his reply, Targoff[5] argues that for large structures it is difficult to simulate free-end conditions and the hypothesis that the measured modes have their errors apportioned in accordance to their modal frequencies is still unproven, especially for modes which occur in groupings with narrow frequency band.

Cannot both sides be right?

In this Note an optimal method is proposed in which the approach given in Ref. 6 is modified so that the rigid body modes are not corrupted and preference can be given, if desired, to groupings of mode shapes.

Let $R(n\times r)$ be a matrix which represents all measured or known mode shapes that have already been selected and orthogonalized. Hence

$$R^tMR=I \tag{1}$$

where $M(n\times n)$ is a known symmetric positive definite mass matrix. In the beginning, R can represent the rigid body modes.

Let $T(n\times q)$ be a matrix which represents the group of measured modes which are now selected to be orthogonalized. It must be noted that the measured modes T_i have to be normalized in the following way:

$$T_i=\tilde{T}_i(\tilde{T}_i^tM\tilde{T}_i)^{-1/2} \tag{2}$$

where $\tilde{T}_i$ is the mode shape before normalization.

Following Ref. 6, we will look for a matrix $Q(n\times q)$ which satisfies the weighted orthogonality conditions

$$Q^tMQ=I \tag{3}$$

and which minimizes the weighted Euclidean norm

$$\phi=\|N(Q-T)\|=n_{ij}(q_{jk}-t_{jk})n_{il}(q_{lk}-t_{lk}) \tag{4}$$

where the Einstein rule of summation is applied. $N(n\times n)$ is the positive definite symmetric solution of the relation

$$N=M^{1/2} \tag{5}$$

Now, in addition to constraint (3), Q must be also orthogonal to R:

$$Q^tMR=0 \tag{6}$$

Note that Eq. (6) is a new constraint which does not appear in Ref. 6.

Using Lagrange multipliers to incorporate the constraints of Eqs. (3) and (6) into the cost function (4) the following Lagrange function is obtained:

$$H=\phi+\lambda_{il}(q_{ji}m_{jk}q_{kl}-\delta_{il})+2\beta_{is}q_{ji}m_{jp}r_{ps} \tag{7}$$

where δ_{il} is the Kronecker delta. β is a matrix of order $(q\times r)$ and Λ is a matrix of order $(q\times q)$. Due to the symmetry of Eq. (3), Λ must be symmetric

$$\Lambda^t=\Lambda \tag{8}$$

The partial differentiation of Eq. (7) with respect to q_{fg}, where the results are equated to zero, yields equations that q_{fg} have to satisfy when H is minimal

$$\frac{\partial H}{\partial q_{fg}}=2n_{if}n_{il}(q_{lg}-t_{lg})+2\lambda_{gl}m_{fk}q_{kl}+2\beta_{gs}m_{fp}r_{ps}=0 \tag{9}$$

written in matrix form, Eq. (9) becomes

$$\frac{\partial H}{\partial Q}=2M(Q-T)+2MQ\Lambda+2MR\beta^t=0 \tag{10}$$

Multiplication of Eq. (10) by R yields

$$\beta^t=R^tMT \tag{11}$$

It can be seen that β^t represents the deviation from the orthonormality between the already orthogonalized modes R and the selection of measured modes T now treated.

By substitution of Eq. (11) into Eq. (10) one obtains

$$Q[I+\Lambda]=P \tag{12}$$

where $P(n\times q)$ is given by

$$P=T-RR^tMT=[I-RR^tM]T \tag{13}$$

Assuming that $I+\Lambda$ is invertable and using Eq. (3), one finally obtains

$$Q=P(P^tMP)^{-1/2} \tag{14}$$

Note that for vanishing R Eq. (14) is identical to Eq. (14) of Ref. 6 where several techniques for its solution are described. It can be shown[6] that Q, obtained from Eq. (14) by using the positive square root of P^tMP, minimizes the function H in Eq. (7).

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